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Perceptual Learning in Mathematics: The Algebra-Geometry Connection

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Abstract

An important component of expertise is the rapid pickup of complex, task-relevant pattern structure, yet such skills are seldom trained explicitly. We report initial results applying principles of perceptual learning to the processing of structure in mathematics, specifically the connection between graphed functions and their symbolic expressions. Subjects in two experiments viewed graphs of functions and made a speeded, forced choice match from several equations. Training consisted of many short trials of this active classification task involving examples of a function (e.g., sine) subjected to various transformations (e.g., scaling, shifting, reflection). Experiment 1 used *contrastive feedback* -- the graph for a trial was shown superimposed on the canonical function to accentuate transformations. Subjects showed substantial performance gains from 45 minutes of training and transferred to new instances, new function families and a new task. In Experiment 2, with contrastive feedback removed, subjects showed no transfer to new functions. The results indicate the value of perceptual training in producing mathematical expertise and the value of contrastive feedback in particular.

What does it mean to attain mathematical expertise? Instruction most often emphasizes declarative knowledge – facts and concepts. A student may learn, for example, that the function y = Sin x can be generated by a certain construction involving a triangle. Having learned about the sine function, the student may be able to answer certain factual questions and work out problems on a test.

There is more to expertise, however, than facts, concepts and inferences. Suppose we ask the student who is familiar with y = Sin x what the graph of y = Sin (x–2) would look like. Chances are the student will not know immediately. Getting the answer may be an inference process requiring several steps: We can substitute 2 for x to find that the function now crosses the x axis at (2,0) instead of (0,0). If we check a few more points, perhaps the answer would become clear.

How would an expert respond to this query? At a glance, it is intuitive that the "-2" in the function y = Sin (x-2) shifts the whole function rightward on the x axis by two units. If it were y = Sin (x+2), the shift would be to the left. Likewise, y = 3 Sin x amplifies the function along the y axis; y = Sin (4x) compresses along the x; and y = 2–Sin x causes a reflection around the x axis and shifts the whole function upward by two units. The expert detects at a glance the structural relationships in each equation and intuitively knows their meaning in the spatial representation (the graph in Cartesian coordinates).

This kind of expertise is important, and in some tasks, decisive. To be able to look at a plot of data and recognize how it could be approximated by an equation, or to visualize the consequences of changes in an equation for the shape of the function, would seem to be basic to the use of mathematics in science. The student who can work out these connections through factual knowledge and reasoning lags behind one who intuits the relevant patterns at a glance. The scientist's ability to extract relevant structures in both equations and graphs allows her attention and effort to be allocated to the scientific problem at hand -- without having to pause to work through what e^{-x} looks like. These aspects of skilled performance are our concern in this paper.

How these skills arise, in mathematics or in other domains, may appear mysterious. In the examples of sine functions above, we could specifically state to the student, in a lecture or text, all of the transformations mentioned. We might find that even after the student learned to state the facts, classification of new examples would be slow and arduous. Attaining the fluent pattern classification skills of the expert may not come from this kind of instruction. We would say that the students "need experience" and that they will attain greater fluency with time. The same is said to beginning radiologists, instrument pilots, accountants and novices in other domains about the structures and patterns they work with.

The passage of time is not very satisfactory as an explanatory notion. A specific hypothesis about the development of such skills is that they involve perceptual learning (Gibson, 1969). Broadly, perceptual learning is defined as "an increase in the ability to extract information from the environment, as a result of experience and practice with stimulation coming from it" (Gibson, 1969). Perceptual learning is a cornerstone of advanced human performance. In some domains, it leads to competence that appears nearly magical. The magic comes from processes that allow for continuing improvement in the extraction of pattern structure with practice. For example, in 1996, Garry Kasparov, the world champion chess player, defeated Deep Blue, a chess-playing computer that examined 125 million

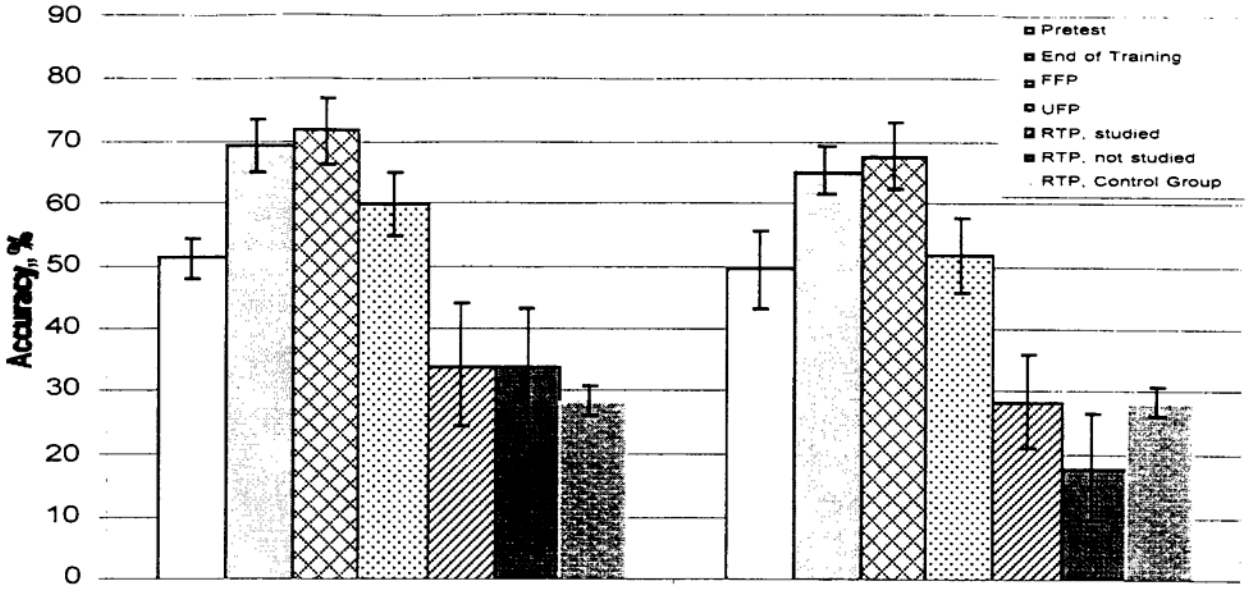

Figure 1. Results for Experiments 1 and 2. Mean accuracy for the different stages of training are shown, including pre-test, end of training, familiar functions post-test (FFP), unfamiliar functions post-test (UFP), remote transfer post-test (RTP) for studied and non-studied functions, and remote transfer post-test for the control group.

12.669, $p<0.001$, and RT, $F(3,54) = 18.596$, $p<0.001$. Individual comparisons showed that participants were significantly better and faster at the end of the training than at the beginning (accuracy: $t(19) = 7.241$, $p<0.001$; reaction time: $t(19) = 7.486$, $p<0.001$), and in the UFP (accuracy: $t(19) = 2.582$, $p<0.02$; reaction time: $t(19) = 4.174$, $p<0.001$). Accuracy was higher in FFP than in both the pre-test, $t(19) = 4.547$, $p<0.001$, and the UFP, $t(19) = 2.239$, $p<0.05$. RTs in the FFP were faster than in the pretest, $t(19) = 4.098$, $p<0.001$, but not reliably different from UFP. Subjects' UFP performance was superior to pretest performance (accuracy: $t(19) = -2.281$, $p< 0.05$; reaction time: $t(19) = 2.666$, $p<0.015$). Accuracy did not differ between EOT and FFP, but the former phase showed better reaction time, $t(19) = 4.198$, $p<.001$. There was a reliable interaction between function trained and condition, $F(3,54)=3.226$, $p<0.03$. For reasons that are unclear, a somewhat larger training effect was found with sines and exponentials than with cosines and logarithmic functions.

The control group did not differ from the pretest group in accuracy, $t(38)=-0.453$, n.s. UFP was faster than the control group ($t(38)=-3.551$, $p<.001$) but not more accurate ($t(38)=1.425$, n.s.). Reaction times in the control group were reliably slower than in the pretest, mean difference = 4.6 sec, $t(38)=-2.661$, $p<0.011$. This result suggests some rapid improvements in response time during the early training trials. (Note that in order to get 10 pretest trials with each function type studied, the pretest comprised the first 10 trials of each of the first two trial blocks, i.e., trials 1-10 and trials 21-30).

Remote Transfer Post-test Results. Performance on the two forms containing combination functions did not differ depending on the function families seen during training. This observation was verified by the analyses. Two 2 X 2 (functions trained by functions tested) ANOVAs, one for accuracy and another for reaction time, yielded no significant main effects or interactions. Comparisons with the control group revealed no reliable differences in terms of accuracy. Response times were quite long for the RTP, on the order of three minutes per form. Control subjects were reliably slower than Experiment 1 subjects for non-studied functions, $t(38)=-2.19$, $p<0.035$, and marginally slower for the studied functions post-test, $t(38) =-1.802$, $P<0.05$ (one tailed).

Discussion

The results of Experiment 1 support several conclusions. First, training designed specifically to foster perceptual learning can improve subjects' performance in relating graphs and equations. Although our subjects had previously learned about the relevant functions in mathematics classes, they had not become skilled in classifying patterns and recognizing transformations. A relatively brief intervention substantially improved both accuracy and speed. A second finding was that training generalized to new instances in the same function families and also to similar transformations deployed in new function families. Training in this experiment did not lead to reliable effects on our remote transfer (combination function) test, which proved quite difficult for subjects.

It does not appear that the 45 minutes of training were sufficient to achieve automatic pattern recognition. At the end of training, accuracy had not reached ceiling. Still, response times had fallen to about 5.5 sec on average. Response times on this order suggest that subjects were becoming more automatic in their pattern classification

rather than using an elaborate reasoning process. Further training might lead to greater automaticity in classification.

What did subjects learn? The transfer results suggest that the training effects were not merely about the trained functions, such as sines. Transformations such as compression, scaling and shifting have the same sorts of effects across function classes. Initial performance levels of subjects suggested that these generalities have not been well learned from conventional mathematics instruction. Our results showed enhancement of performance after training even with new function families, suggesting that the symbolic and graphical "meanings" of basic transformation patterns were learned to some degree. The results suggest that PLMs may have great promise for developing fluent pattern processing in mathematics.

Experiment 2: Contrastive Feedback

Experiment 1 indicated the efficacy of perceptual classification training, using several ingredients suggested by earlier research and by intuition. Which ingredients are crucial to the usefulness of this kind of training? These questions have hardly begun to be addressed, especially in the application of perceptual learning to complex skills, such as doing mathematics. A goal of our research is to determine systematically the effects of particular aspects of training in order to optimize PLMs in education.

In Experiment 2, we examined whether the particular kind of contrastive feedback used in Experiment 1 had important effects on learning. Recall that after each problem, participants viewed the problem function superimposed on the canonical function. If this particular type of feedback facilitated discovery of relevant pattern transformations, then eliminating it might reduce the success of training. In particular, we hypothesized that this type of feedback might have been especially helpful in producing transfer to new function families. Eliminating it might therefore be expected to reduce or eliminate transfer of learning to new functions.

Method

Participants. Twenty undergraduate students at the University of California, Los Angeles, received credit units for participation.

All aspects of the method were identical to Experiment 1 except that contrastive feedback was eliminated. Instead, feedback screens indicated whether the response on the trial had been correct and displayed the graph and correct equation for that problem.

Results

Training, FFP and UFP Results. As in experiment 1, training produced highly reliable effects on accuracy (shown in Figure 1) and RT. Subjects improved from about 50% correct in the pretest to about 65% in the final block of training. Response times decreased about 40% during training. Both the accuracy and speed at the end of training were maintained in the FFP. In contrast to the results of Experiment 1, there was little or no transfer of training to the UFP; training effects were largely confined to the familiar (studied) function types. The control group, pretest and UFP performance of the training group were very similar.

These observations were confirmed by the analyses. Accuracy and RTs were each analyzed using a 2 (functions trained) X 4 (phase: Pretest, EOT, FFP, UFP) ANOVA, with repeated measures on the latter factor. There were reliable main effects of phase, for accuracy $F(3,54) = 8.892$, $p<0.001$, and for RT, $F(3,54) = 8.405$, $p<0.001$. Individual comparisons showed that participants were significantly better and faster at EOT than at the beginning (for accuracy: $t(19) = 3.507$, $p<0.002$; for reaction time: $t(19) = 3.816$, $p<0.001$). Accuracy and reaction time were better in FFP than in the pre-test (accuracy: $t(19) = 3.789$, $p<0.001$; reaction time: $t(19) = 3.165$, $p<0.005$) and also better than in the UFP (accuracy: $t(19) = 3.47$, $p<0.003$; reaction time: $t(19) = 2.687$, $p<0.015$). Accuracy did not reliably differ between EOT and FFP, $t(19) = 0.76$, n.s. For reaction time, there was a marginal advantage for the EOT problems over FFP, $t(19) = 1.96$, $.05<p<0.10$. Subjects' UFP performance did not differ significantly from the pretest (accuracy: $t(19) = 0.453$, n. s.; reaction time: $t(19) = 0.805$, n. s.).

Accuracy in the control group did not differ from either the pretest ($t(38) = -0.59$, n.s.) or the UFP ($t(38) = -0.192$, n.s). Reaction times in the control group were significantly slower than in both the pretest, $t(38) = -2.706$, $p<0.01$, and UFP, $t(38) = -3.359$, $p<.002$.

Remote Transfer Post-test Results. As in Experiment 1, performance on the two forms containing combination functions did not differ depending on the trained function families. Two 2 X 2 (functions trained by functions tested) ANOVAs, one for accuracy the other for reaction time, yielded no significant main effects or interactions.

Accuracy and reaction times on both forms (studied and unstudied functions) of the RTP did not differ reliably from a control group that received no training.

Cross-Experiment Comparisons

To more carefully assess the effect of contrastive feedback, we performed several analyses including the data from both Experiments 1 and 2.

Training, FFP and UFP Results. ANOVAs were performed on accuracy and reaction time data. Each was a 2 X 2 X 4 (experiment by functions trained by condition) design with repeated measures on the condition factor. There was a strong main effect of condition (accuracy: $F(3, 108) = 20.56$, $p <.001$; RT: $F(3,108) = 21.99$, $p <.001$), indicating the effects of training in both experiments. There was also a condition by functions trained interaction for accuracy, $F(3, 108) = 4.188$, $p <.008$, indicating that in both experiments training effects were greater for subjects trained on sines and exponentials. No other main effects or interactions reached significance in the accuracy or RT analyses.

The lack of a reliable experiment by condition interaction for accuracy contrasts somewhat with a difference between

Experiment 1 and 2 observed above. Specifically, in Experiment 1, participants showed better accuracy on unfamiliar functions (UFP) than in the pretest, $t(19) = -2.281$, $p < 0.05$, whereas no such difference appeared in Experiment 2, $t(19) = 0.453$, n.s.

This difference suggests that contrastive feedback produced better learning of transformations that could be applied to new function families. The lack of direct support for this interaction in the combined analysis is probably due to the greater variability present in between-subjects comparisons. Nevertheless, the mixed results suggest caution. Fortunately, the cross-experiment comparisons of RTP performance (below) provide some confirmation of the difference in training effects between the two experiments.

Remote Transfer Post-test Results. Remote transfer effects of training appeared to be somewhat larger in Experiment 1, which used contrastive feedback, than in Experiment 2, which did not. Even though no reliable differences were found between the studied and non-studied functions within each experimental condition, a difference was found in terms of accuracy when comparing experiments 1 and 2. More specifically, subjects in experiment 1 performed better than subjects in experiment 2 for the non-studied functions ($t(38)=1.906$, $p<.05$, one tailed). The superior performance of experiment 1 subjects' in the non-studied functions attests for the effect of the contrastive feedback in producing better learning of transformations that can be applied to new function families. No other comparisons reached significance.

General Discussion

Taken together, the results of experiments 1 and 2 indicate a clear training effect. Subjects in both experiments substantially improved their accuracy and speed in relating graphs and equations, as compared to a control group. Subjects' improvement extended beyond the specific examples on which they were trained. They showed equally good performance on new examples from the function families on which they had trained, indicating that learning did not depend merely on memorizing specific instances.

Subjects in Experiment 1 also transferred their learning to *new* function families. This result did not appear in Experiment 2, however. These outcomes suggest that the *contrastive feedback* used in Experiment 1 may have laid the foundation for transfer by directing attention to the relevant *transformations* in the stimulus patterns. The results are consistent with the idea that perceptual learning might be accelerated by augmented feedback that helps to direct attention to relevant features and dimensions.

Our remote transfer test was difficult and showed modest effects. Here again, however, a comparison between the two experiments revealed that Experiment 1 produced more gains in accuracy for the non-studied functions than Experiment 2, which did not have contrastive feedback.

The contributions of other procedural ingredients remain to be assessed. For example, it is often asserted that perceptual learning depends on an active classification task (e.g., Karni & Sagi, 1993), but there have been few careful tests of this idea. In the present work, both groups were given active classification tasks. Currently we are studying whether mere exposure in short episodes to corresponding graphs and equations produces learning and transfer.

In summary, our results indicate that a brief period of perceptual training can substantially improve subjects' performance in processing mathematical structures expressed in graphs and equations. Contrastive feedback – highlighting the dimensions of difference between a basic function and transformations of it -- enhances the learning of relationships that transfer to novel function families. Perceptual learning modules may have great promise in accelerating the development of components of expertise that do not arise easily from traditional instruction.

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